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CREEP AND STRESS RELAXATION OF AN INCOMPRESSIBLE VISCOELASTIC MATERIAL OF THE RATE TYPE

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Abstract—In this work, a constitutive equation for an isotropic, nonlinear viscoelastic material of the rate type is introduced to study the nonlinear viscoelastic responses of rubber-like materials. This constitutive equation not only predicts creep, recovery, and stress relaxation processes which are of significant interest to engineering applications but also depicts these processes through a simple mathematical structure. The constitutive theory of the rate type generalizes the standard linear solid of classical linear viscoelasticity and contains the viscoelasticity of the differential type as its special case. It has been found that the stress relaxation process is characterized by a universal solution regardless of the response functions of the material. For the creep process, the only difference existing between the viscoelastic materials of the rate type and the differential type is the creeping speed. Closed form solutions are obtained for creep of a viscoelastic Mooney–Rivlin material in simple shear and a viscoelastic neo-Hookean material in simple extension.

1. INTRODUCTION

In a recent study (Beatty and Zhou, 1991), a viscoelastic constitutive equation of the differential type, a class which includes the Voigt–Kelvin solid of classical linear viscoelasticity, was used to study the nonlinear response of the material in simple shear deformation. This constitutive equation predicts the creep and recovery processes observed in viscoelastic materials and provides analytical solutions to the finite amplitude oscillations of a load supported by prestretched shear mountings of viscoelastic Mooney–Rivlin materials of the differential type. It has been found that the primary homogeneous prestretch plays an important role in determination of all aspects of the mechanical response. Lack of the stress relaxation character typical of rubber materials, however, is a major shortcoming of this theory.

It is well known that the constitutive equation of the differential type is one of the three basic types of constitutive theories for simple materials (Truesdell and Noll, 1965). The other two are the rate type and the integral type. The integral type theory, while being widely used in engineering applications, was first developed from the fact that the viscoelastic material exhibits the property of hereditary response. That is, the present state of stress depends not only upon the present state of deformation, but also upon previous states. Typical examples include Boltzmann's theory, Leitman and Fisher's theory (1973) for infinitesimal deformations, Green's multiple integral representation (Green and Rivlin, 1957, 1959; Green *et al.*, 1959), and the BKZ models (Bernstein *et al.*, 1963) for finite deformations. However, mathematical complexity is its main drawback. In this work, I shall present a special constitutive equation of the rate type and explore its engineering applications in predicting the stress relaxation process. The effects on the creep and recovery processes of this constitutive model compared with the differential type theory will also be addressed.

An explicit form of the constitutive equation for a class of incompressible, isotropic viscoelastic materials of the rate type will be described in Section 2. This constitutive equation contains the differential type model established by Beatty and Zhou (1991) as its special case and generalizes the standard linear solid of classical linear viscoelasticity. The nonlinear theory is then applied in Section 3 to study the stress relaxation process. A universal solution independent of the response functions of the materials is obtained. The creep and recovery processes are discussed in Section 4 and the results are compared with the solutions obtained from the differential type theory. The analysis shows that the only difference between these two theories is the creeping speed.

2. A CONSTITUTIVE EQUATION OF THE RATE TYPE

In this section, an explicit form of the constitutive equation of the rate type will be given. This constitutive equation contains the viscoelastic material of the differential type as its special case and generalizes the standard linear solid of classical linear viscoelasticity. I shall begin with a brief review of the kinematics of continuum mechanics and the constitutive equations of hyperelastic solid, Newtonian fluid, and viscoelastic materials of the differential type.

2.1. *Preliminaries*

We consider a body in a Euclidean space of three dimensions to undergo a deformation described by

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t), \tag{1}$$

where $\mathbf{x}$ and $\mathbf{X}$ are the respective position vectors of a typical particle of the body at an arbitrary time t and a reference time t_R. We recall the deformation gradient $\mathbf{F}$, the Cauchy–Green deformation tensor $\mathbf{B}$, and the spatial velocity gradient tensor $\mathbf{L}$, defined by

$$\mathbf{F} \equiv \frac{\partial \mathbf{x}(\mathbf{X}, t)}{\partial \mathbf{X}}, \quad \mathbf{B} \equiv \mathbf{F}\mathbf{F}^T, \quad \mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}, \tag{2}$$

where the superimposed dot denotes the usual material time derivative. We also recall the stretching tensor $\mathbf{D}$ given by

$$\mathbf{D} = \tfrac{1}{2}(\mathbf{L} + \mathbf{L}^T). \tag{3}$$

An isotropic and incompressible, hyperelastic solid is a material whose constitutive equation is given by

$$\mathbf{T} = -p\mathbf{1} + \beta_1 \mathbf{B} + \beta_{-1}\mathbf{B}^{-1}, \tag{4}$$

where $\mathbf{T}$ is the Cauchy stress tensor and p is the undetermined pressure due to the incompressibility constraint. The response functions are given by

$$\beta_1 = 2\frac{\partial \Sigma}{\partial I_1}, \quad \beta_{-1} = -2\frac{\partial \Sigma}{\partial I_2}, \tag{5}$$

where I_i $(i = 1, 2)$ are the principal invariants of $\mathbf{B}$ and Σ is the strain energy function per unit volume in the reference configuration.

A Mooney–Rivlin material is an incompressible material whose strain energy function is a linear function of the first and the second invariants of $\mathbf{B}$ (Beatty, 1987). The strain energy function for this model is given by

$$\Sigma = \frac{G}{2(1+\alpha)}[(I_1 - 3) + \alpha(I_2 - 3)]. \tag{6}$$

Hence, by (6), eqns (4) and (5) yield

$$\beta_1 = \frac{G}{1+\alpha}, \quad \beta_{-1} = -\frac{\alpha G}{1+\alpha}, \tag{7}$$

$$\mathbf{T} = -p\mathbf{1} + \frac{G}{1+\alpha}[\mathbf{B} - \alpha\mathbf{B}^{-1}], \tag{8}$$

where G is the shear modulus and α is a positive material parameter, usually between 0 and 1. When $\alpha = 0$, the Mooney–Rivlin model reduces to the well-known neo-Hookean model which was obtained from statistical mechanics (Treloar, 1975). The strain energy function and the constitutive equation for the neo-Hookean model can be written as:

$$\Sigma = \frac{G}{2}(I_1 - 3), \tag{9}$$

$$\beta_1 = G, \quad \beta_{-1} = 0, \tag{10}$$

$$\mathbf{T} = -p\mathbf{1} + G\mathbf{B}. \tag{11}$$

The Mooney–Rivlin and the neo-Hookean models will be used in the study of creep and recovery processes in Section 4. In addition, according to Gurtin (1981), the constitutive equation for a Newtonian fluid is given by

$$\mathbf{T} = -p\mathbf{1} + 2\eta\mathbf{D}, \tag{12}$$

where η is the viscosity of the fluid. Since the Newtonian fluid is an incompressible fluid, we have $I_1(\mathbf{D}) = \mathrm{tr}\,\mathbf{D} = 0$.

A constitutive equation for viscoelastic and incompressible materials of the differential type is studied by Beatty and Zhou (1991) and is given by

$$\mathbf{T} = -p\mathbf{1} + \beta_1\mathbf{B} + \beta_{-1}\mathbf{B}^{-1} + 2\eta\mathbf{D}. \tag{13}$$

When $\eta = 0$, eqn (13) yields the familiar constitutive equation for an incompressible, isotropic elastic solid (4). By comparing (4) and (12) we realize that the constitutive equation (13) describes the uncoupled linear viscous and nonlinear elastic response of an isotropic, incompressible material. For brevity, we call the material described by (13) the viscoelastic material of the differential type.

2.2. *A viscoelastic constitutive equation of the rate type*

Truesdell and Noll (1965) indicate that the general constitutive equation of the rate type for an isotropic material has a form of

$$\overset{\dagger}{\mathbf{T}}_q = \mathscr{R}(\mathbf{T}, \overset{\dagger}{\mathbf{T}}, \ldots, \overset{\dagger}{\mathbf{T}}_{q-1}; \mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_r; \mathbf{B}), \tag{14}$$

where $\mathbf{A}_i$ is the Rivlin–Ericksen tensor with $\mathbf{A}_1 = 2\mathbf{D}$ and $\overset{\dagger}{\mathbf{T}}_i$ is the ith convected stress rate with

$$\overset{\dagger}{\mathbf{T}}_1 = \overset{\dagger}{\mathbf{T}} = \dot{\mathbf{T}} + \mathbf{L}^T\mathbf{T} + \mathbf{T}\mathbf{L}. \tag{15}$$

Let us consider a special case of (14) in which $q = 1$ and $r = 1$. Hence, eqn (14) reduces to

$$\overset{\dagger}{\mathbf{T}} = \mathscr{I}(\mathbf{T}, \mathbf{D}, \mathbf{B}). \tag{16}$$

We now consider a subclass of (16) where the response function $\mathscr{I}$ is a polynomial of $\mathbf{T}$,

D, and **B**. Particularly, we shall examine a special class of incompressible materials whose constitutive equation (16) is in the form of

$$\dot{\mathbf{T}}+\mathbf{L}^T\mathbf{T}+\mathbf{T}\mathbf{L} = -\phi_1[\mathbf{T}-(-p\mathbf{1}+\beta_1\mathbf{B}+\beta_{-1}\mathbf{B}^{-1}+2\eta\mathbf{D})]. \tag{17}$$

The response functions in (17) carry the same meaning as those in (13). ϕ_1 is a material constant. When $\phi_1 = 0$, eqn (17) reduces to

$$\dot{\mathbf{T}}+\mathbf{L}^T\mathbf{T}+\mathbf{T}\mathbf{L} = 0 \tag{18}$$

or

$$\dot{\mathbf{T}} = -\mathbf{L}^T\mathbf{T}-\mathbf{T}\mathbf{L}. \tag{19}$$

Equation (19) is a special case of the general constitutive equation for a hypoelastic material (Truesdell and Noll, 1965, Section 99). On the other hand, if (19) holds, by eqn (17), we have either the constitutive equation (13) for viscoelastic material of the differential type or $\phi_1 = 0$. For brevity, we shall call the material described by (17) the viscoelastic material of the rate type.

2.3. *Relation to classical linear viscoelasticity*

The linearized infinitesimal theory of (17) may be obtained through the following relations from continuum mechanics. If we let $\mathbf{F} = \mathbf{1}+\mathbf{G}$, where $\mathbf{G}$ is the usual infinitesimal deformation gradient, and recall the infinitesimal engineering strain $\boldsymbol{\varepsilon} = \frac{1}{2}(\mathbf{G}+\mathbf{G}^T)$, we find by (2) that

$$\mathbf{B} = \mathbf{1}+2\boldsymbol{\varepsilon}, \quad \mathbf{D} = \dot{\boldsymbol{\varepsilon}}. \tag{20}$$

In deriving (20), all products of $\mathbf{G}$ and $\dot{\mathbf{G}}$ have been neglected. Hence, to the first order in $\boldsymbol{\varepsilon}$ and $\dot{\boldsymbol{\varepsilon}}$, the constitutive equation (17) is approximated by

$$\dot{\mathbf{T}}+\phi_1\mathbf{T} = \phi_1(-\bar{p}\mathbf{1}+2G\boldsymbol{\varepsilon}+2\eta\dot{\boldsymbol{\varepsilon}}), \tag{21}$$

where $\mathbf{T}$ is now the same as the engineering stress tensor and $\bar{p}$ is another arbitrary, undetermined hydrostatic pressure given by

$$\bar{p} = p-(\beta_1+\beta_{-1}) \tag{22}$$

and the shear modulus G given by

$$G = \beta_1-\beta_{-1}. \tag{23}$$

Equation (21) shows the linear relation among the stress rate, the stress, the strain, and the strain rate and hence is recognized as the constitutive equation for the familiar incompressible standard linear solid of classical linear viscoelasticity. Hence, material of (17) is a special kind of generalized incompressible standard linear solid for finite deformations.

3. UNIVERSAL SOLUTION FOR STRESS RELAXATION PROCESS

The stress relaxation phenomenon has been observed in all viscoelastic materials and is characterized by the decay of stress under certain constant deformation. For viscoelastic material of the rate type (17), it will be shown that the stress relaxation process is characterized by a universal solution regardless of the response functions of the material.

To begin with the stress relaxation process, we look at the equilibrium position of (17) which is given by

$$\hat{\mathbf{T}} = -\hat{p}\mathbf{1}+\hat{\beta}_1\hat{\mathbf{B}}+\hat{\beta}_{-1}\hat{\mathbf{B}}^{-1}, \tag{24}$$

where a circumflex denotes the values at equilibrium position in which all the quantities are time-independent constants. It is clear that the stress and corresponding deformation at equilibrium position can be determined completely by (4), the constitutive equation for incompressible hyperelastic materials.

Stress relaxation is a decay process of decreasing stress $\mathbf{T}$ under constant deformation $\hat{\mathbf{B}}$. Hence, the equation for the stress relaxation process is found from (17) as

$$\dot{\mathbf{T}} = -\phi_1[\mathbf{T}-\hat{\mathbf{T}}], \tag{25}$$

where the constant deformation is reflected through the constant equilibrium stress $\hat{\mathbf{T}}$ given by (24). Equation (25) can be written in the form of

$$\frac{\mathrm{d}T_{ij}}{\mathrm{d}t} = -\phi_1[T_{ij}-\hat{T}_{ij}] \tag{26}$$

with its solution given by

$$T_{ij} = \hat{T}_{ij}+[T^0_{ij}-\hat{T}_{ij}]\mathrm{e}^{-\phi_1 t}, \tag{27}$$

where T^0_{ij} is the initial stress for the relaxation process.

To obtain a physically meaningful result for the relaxing stress T_{ij}, the material constant ϕ_1 must be positive. In this case, the stress relaxation process of the viscoelastic material of the rate type (17) starts from certain initial stress T^0_{ij}, relaxes in an exponential way, and finally approaches the equilibrium state $\hat{T}_{ij}$ determined by the elasticity theory. This solution is valid for all viscoelastic materials of the rate type (17). It is indepedent of the material constants and the specific forms of the deformation and is hence a universal solution.

The universal solution (27) also shows the important physical information the constant ϕ_1 carries. It reflects the speed of the relaxation process. Theoretically, it takes an infinitely long time to reach the equilibrium state. On the other hand, the stress in question apparently relaxes with a nonuniform speed. Most parts of the process are achieved within a relatively short period of time. More precisely, we recall that in the viscoelasticity literature, the term retardation time (t_r) is often used as a measure of this property. The retardation time in the present situation is defined as

$$t_r \equiv \frac{1}{\phi_1}. \tag{28}$$

By (27), the ratio $(T^0_{ij}-T_{ij})/(T^0_{ij}-\hat{T}_{ij})$ at $t = t_r$ determines the constant retardation ratio

$$\frac{T^0_{ij}-T_{ij}}{T^0_{ij}-\hat{T}_{ij}} = 1-\mathrm{e}^{-1} \approx 0.632. \tag{29}$$

This is a universal constant of the stress relaxation process for all viscoelastic materials of (17). This universal constant is the same as the one obtained by Beatty and Zhou (1991) in the study of creep process of the viscoelastic Mooney–Rivlin material of differential type in simple shear. Physically, relation (29) shows that 63.2% of the total stress relaxation process has been accomplished by the time $t = t_r$.

This completes the analysis of the stress relaxation process of the viscoelastic material of the rate type. We next consider the creep and recovery processes and examine the difference between the rate type theory and the differential type theory.

4. CREEP AND RECOVERY PROCESSES OF THE VISCOELASTIC MATERIAL OF THE RATE TYPE

We recall that creep is the growth process of deformation under constant stress to an asymptotic or ultimate equilibrium state and recovery is a decay process marked by decreasing deformation from an arbitrary state at which the stress is suddenly removed or reduced to a lesser value. In either case, the governing equation is obtained from (17) and is given by

$$\mathbf{L}^T\hat{\mathbf{T}}+\hat{\mathbf{T}}\mathbf{L} = -\phi_1[\hat{\mathbf{T}}-(-p\mathbf{1}+\beta_1\mathbf{B}+\beta_{-1}\mathbf{B}^{-1}+2\eta\mathbf{D})], \tag{30}$$

where $\hat{\mathbf{T}}$ is the constant symmetric stress tensor. By (3), the creep or recovery equation is reduced to

$$(\hat{\mathbf{T}}-\phi_1\eta\mathbf{1})\mathbf{L}+\mathbf{L}^T(\hat{\mathbf{T}}-\phi_1\eta\mathbf{1}) = -\phi_1[\hat{\mathbf{T}}-(-p\mathbf{1}+\beta_1\mathbf{B}+\beta_{-1}\mathbf{B}^{-1})]. \tag{31}$$

This is a first order differential equation for deformation. Its solution certainly depends on the response functions of the material. We shall examine their closed form solutions for two cases: the creep and recovery processes of a Mooney–Rivlin material in simple shear and the creep and recovery processes of a neo-Hookean material in simple extension. Before we get into the detailed analysis for each case, it is interesting to point out that the creep equation (31) may be written in the form of

$$\mathbf{Z}+\mathbf{Z}^T = -\phi_1[\hat{\mathbf{T}}-\mathbf{T}_H], \tag{32}$$

where $\mathbf{T}_H$ is the instantaneous hyperelastic stress tensor corresponding to the creeping deformation and has a form of (4). Tensor $\mathbf{Z}$ is defined by

$$\mathbf{Z} \equiv (\hat{\mathbf{T}}-\phi_1\eta\mathbf{1})\mathbf{L}. \tag{33}$$

For the following creep and recovery analysis, it is also useful to write the constitutive equation (17) in the form of

$$\mathbf{T} = -p\mathbf{1}+\beta_1\mathbf{B}+\beta_{-1}\mathbf{B}^{-1}+2\eta\mathbf{D}-\frac{1}{\phi_1}(\dot{\mathbf{T}}+\mathbf{L}^T\mathbf{T}+\mathbf{T}\mathbf{L}). \tag{34}$$

It is clear that eqn (34) contains the constitutive equation (13) for the viscoelastic material of the differential type as its special case. When $\phi_1 \to \infty$, eqn (34) reduces to (13). Hence, the solutions of creep and recovery processes of the differential type theory may be obtained from that of the rate type theory by letting $\phi_1 \to \infty$.

4.1. *Creep and recovery in simple shear*

For simple shear deformation, we consider a rigid body of mass M on a smooth surface making an angle θ with the horizontal plane supported symmetrically between identical, prestretched rubber springs of original length L and cross-sectional area A. The springs, prestretched an amount λ_0, are bounded to the body at one end and to rigid end supports

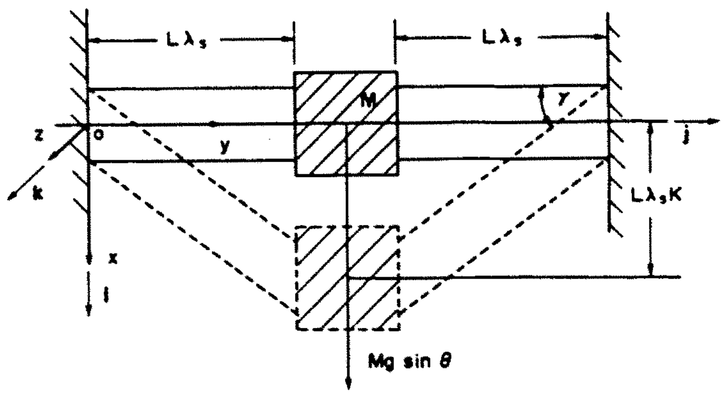

Fig. 1. A rigid body M supported symmetrically between identical prestretched viscoelastic rubber shear springs.

at the other, as shown in Fig. 1. We suppose that each rubber spring executes an ideal, time-dependent simple shearing deformation of amount $K(t)$ superimposed on the static longitudinal stretch λ_s. Certainly, the simple shear is an ideal deformation. Though bending will occur with the shearing, we shall ignore the bending effect for mathematical simplicity. We consider an incompressible viscoelastic model characterized by (34). Hence, for the left spring, the motion may be defined by the following rectangular Cartesian coordinate relation for the present place (x, y, z) occupied by the particle whose place was at (X, Y, Z) initially:

$$x = \lambda_s^{-1/2}X + K(t)\lambda_s Y, \quad y = \lambda_s Y, \quad z = \lambda_s^{-1/2}Z. \tag{35}$$

Let **i**, **j**, and **k** denote the usual rectangular Cartesian basis in the directions of x, y, z, respectively, as shown in Fig. 1. Hence, by (35) and (2) we find

$$\mathbf{F} = \lambda_s^{-1/2}(\mathbf{i}\otimes\mathbf{i}+\mathbf{k}\otimes\mathbf{k})+\lambda_s\mathbf{j}\otimes\mathbf{j}+K\lambda_s\mathbf{i}\otimes\mathbf{j}, \tag{36}$$

$$\mathbf{B} = (\lambda_s^{-1}+K^2\lambda_s^2)\mathbf{i}\otimes\mathbf{i}+\lambda_s^2\mathbf{j}\otimes\mathbf{j}+\lambda_s^{-1}\mathbf{k}\otimes\mathbf{k}+K\lambda_s^2(\mathbf{i}\otimes\mathbf{j}+\mathbf{j}\otimes\mathbf{i}), \tag{37}$$

$$\mathbf{B}^{-1} = \lambda_s(\mathbf{i}\otimes\mathbf{i}+\mathbf{k}\otimes\mathbf{k})+(\lambda_s^{-2}+K^2\lambda_s)\mathbf{j}\otimes\mathbf{j}-K\lambda_s(\mathbf{i}\otimes\mathbf{j}+\mathbf{j}\otimes\mathbf{i}), \tag{38}$$

$$\mathbf{L} = \dot{K}\mathbf{i}\otimes\mathbf{j}, \tag{39}$$

$$\mathbf{D} = 1/2\dot{K}(\mathbf{i}\otimes\mathbf{j}+\mathbf{j}\otimes\mathbf{i}), \tag{40}$$

$$I_1(\mathbf{B}) = \lambda_s^2+2\lambda_s^{-1}+K^2\lambda_s^2, \quad I_2(\mathbf{B}) = 2\lambda_s+\lambda_s^{-2}+K^2\lambda_s, \quad I_3(\mathbf{B}) = 1, \tag{41}$$

$$I_1(\mathbf{D}) = 0, \quad I_2(\mathbf{D}) = -1/4\dot{K}^2. \tag{42}$$

With the aid of the foregoing relations, we find by (34)

$$T_{11} = -p+\beta_1(\lambda_s^{-1}+K^2\lambda_s^2)+\beta_{-1}\lambda_s-\frac{1}{\phi_1}\dot{T}_{11}, \tag{43}$$

$$T_{22} = -p+\beta_1\lambda_s^2+\beta_{-1}(\lambda_s^{-2}+K^2\lambda_s)-\frac{1}{\phi_1}(\dot{T}_{22}+2\dot{K}T_{12}), \tag{44}$$

$$T_{33} = -p+\beta_1\lambda_s^{-1}+\beta_{-1}\lambda_s-\frac{1}{\phi_1}\dot{T}_{33}, \tag{45}$$

$$T_{12} = \lambda_s K(\lambda_s \beta_1 - \beta_{-1}) + \eta \dot{K} - \frac{1}{\phi_1}(\dot{T}_{12} + \dot{K} T_{11}). \tag{46}$$

$$T_{13} = -\frac{1}{\phi_1}\dot{T}_{13}. \tag{47}$$

$$T_{23} = -\frac{1}{\phi_1}(\dot{T}_{23} + \dot{K} T_{13}). \tag{48}$$

The combination of eqns (43), (44), and (46) generates the following relation between the shear stress and the material constants η and ϕ_1:

$$T_{11} - T_{22} = \left[T_{12} - \eta \dot{K} + \frac{1}{\phi_1}(\dot{T}_{12} + \dot{K} T_{11})\right]\frac{\lambda_s^{-1} + \lambda_s^2(K^2 - 1)}{K\lambda_s^2} - \frac{1}{\phi_1}[\dot{T}_{11} - \dot{T}_{22} - 2\dot{K} T_{12}]. \tag{49}$$

Hence, at equilibrium state for which all terms involving the time derivative vanish, we obtain the well-known universal relation in simple shear.

Strictly speaking, bending will occur along with the shear deformation. In order to obtain simple shear deformation, surface traction corresponding to T_{11} has to apply to keep the rubber spring from bending (see Beatty and Zhou, 1991). However, we still have the surface traction-free condition of $T_{33} = 0$. This yields

$$-p = -\beta_1 \lambda_s^{-1} - \beta_{-1} \lambda_s. \tag{50}$$

Hence, we find

$$T_{11} = \beta_1 K^2 \lambda_s^2 - \frac{1}{\phi_1}\dot{T}_{11}. \tag{51}$$

For simple shear deformation, the creep process is characterized by growth of $K(t)$ under the constant stresses of $T_{12} = \hat{T}_{12}$ and $T_{11} = \hat{T}_{11}$, say. These constant stresses can be executed by the spring–mass system shown in Fig. 1 where $2A\hat{T}_{12} = Mg \sin\theta$ and $\hat{T}_{11}$ is balanced due to the symmetry of the spring–mass system. We expect that if the load M is released when $K = 0$, the shear will increase asymptotically to an ultimate equilibrium state defined by $\dot{K}(t) \to 0$ and $K(t) \to K_s$ as $t \to \infty$. Hence, by (46) and (51), the static equilibrium shear deflection K_s is related to $\hat{T}_{12}$ and $\hat{T}_{11}$ through

$$\hat{T}_{12} = \lambda_s K_s[\lambda_s \hat{\beta}_1 - \hat{\beta}_{-1}], \tag{52}$$

$$\hat{T}_{11} = \hat{\beta}_1 K_s^2 \lambda_s^2, \tag{53}$$

where $\hat{\beta}_1$ and $\hat{\beta}_{-1}$ are functions of λ_s and K_s alone and hence are constants independent of time t. We also recall that the recovery phenomenon is a decay process marked by decreasing shear $K(t)$ from an initially deformed state following a sudden reduction in the applied force. In particular, if the process begins from the static state determined by (52) and (53) and the force is reduced to zero, we expect that the recovery shear $K(t)$ decreases asymptotically from K_s to zero. This will never happen in real materials since creep is an irreversible process. For real materials, energy dissipates while creeping. However, creep and recovery are slow motions and this particular material is actually an elastic material for static problems. Hence, when the stress reduces to zero, the material returns to its original undeformed state. This is true for both simple shear and simple extension.

The governing equations for creep and recovery can be obtained from (46). We find for creep

$$\dot{K}\left(\eta-\frac{1}{\phi_1}\hat{T}_{11}\right)=\hat{T}_{12}-\lambda_s K(\lambda_s\beta_1-\beta_{-1}) \tag{54}$$

and for recovery

$$\eta\dot{K}=-\lambda_s K(\lambda_s\beta_1-\beta_{-1}). \tag{55}$$

The right-hand side of eqn (54) is in the same form as eqn (26) obtained by Beatty and Zhou (1991) for creep of the viscoelastic material of the differential type in simple shear. In their work, the creep process is treated in a more general case where the simple shear deformation is superimposed on a triaxial deformation. In the present situation, the simple shear deformation is superimposed on a longitudinal static extension which is a special case in the frame of the work by Beatty and Zhou (1991). Without considering this point, the differential equation (55) is the same as eqn (28) obtained by Beatty and Zhou (1991) for recovery of a viscoelastic material of the differential type in simple shear. In other words, two materials have the same recovery response in simple shear. Hence, the recovery test cannot distinguish one material from the other. We shall refer the detailed discussion on the recovery process to the work by Beatty and Zhou (1991) and focus our attention to the difference of creeping speed predicted by these two models. The difference in creeping speed between the rate type theory and the differential type theory is reflected through the coefficient $\dot{K}$ in (54). The solution to (54) depends on the response functions β_1 and β_{-1}. We shall discuss this creep process for the viscoelastic Mooney–Rivlin material described in Section 2.

By (7), eqn (54) for creep is reduced to

$$\left[\frac{\eta}{G}-\frac{K_s^2\lambda_s^2}{(1+\alpha)\phi_1}\right]\frac{\mathrm{d}K}{\mathrm{d}t}=\frac{\lambda_s(\lambda_s+\alpha)}{1+\alpha}(K_s-K). \tag{56}$$

By comparing the results obtained by Beatty and Zhou (1991) we find that the difference in creep process between these two models is reflected by the retardation time t_r. If we define the retardation time t_r to be

$$t_r=\frac{\eta}{G}\frac{1+\alpha}{(\lambda_s+\alpha)\lambda_s}-\frac{\lambda_s K_s^2}{(\lambda_s+\alpha)\phi_1}, \tag{57}$$

then the closed form solution to (56) is given by

$$K=K_s(1-\mathrm{e}^{-t/t_r}). \tag{58}$$

Solution (58) is in the same form as the result obtained by Beatty and Zhou (1991) for creep of a viscoelastic Mooney–Rivlin material of the differential type. Hence, the only difference between these two theories in this case is the retardation time t_r. It is recognized that the first term in (57) is the retardation time for the differential type theory obtained by Beatty and Zhou (1991). Hence, the rate type theory contains the differential type theory as a special case. Since λ_s, α, and the material constant ϕ_1 are all positive, we conclude from (57) that the retardation time t_r for rate type materials is smaller than that of differential type materials. In other words, the creep process of a rate type material moves faster than that of a differential type material. We also notice from (57) that the retardation time for a rate type material depends on the static shear deflection K_s while t_r for a differential type material is independent of K_s (Beatty and Zhou, 1991).

It is apparent that the ratio K/K_s at $t=t_r$, which is

$$K/K_s=1-\mathrm{e}^{-1}\approx 0.632, \tag{59}$$

determines the constant retardation ratio. In other words, 63.2% of the total creep process has been accomplished by the time $t = t_r$. This constant retardation ratio has been reported by Beatty and Zhou (1991) for the same creep process of a viscoelastic Mooney–Rivlin material of the differential type. We hence conclude our study on creep and recovery of the material in simple shear and move on to the same problem in simple extension.

4.2. Creep and recovery in simple extension

Simple extension is a homogeneous motion. If we let λ be the longitudinal stretch, the deformation for incompressible material is defined by

$$x = \lambda X, \quad y = \lambda^{-1/2} Y, \quad z = \lambda^{-1/2} Z. \tag{60}$$

Hence, by (2), the deformation tensors are found as:

$$[\mathbf{F}] = \operatorname{diag}\{\lambda, \lambda^{-1/2}, \lambda^{-1/2}\}, \tag{61}$$

$$[\mathbf{B}] = \operatorname{diag}\{\lambda^2, \lambda^{-1}, \lambda^{-1}\}, \quad [\mathbf{B}^{-1}] = \operatorname{diag}\{\lambda^{-2}, \lambda, \lambda\}, \tag{62}$$

$$[\mathbf{D}] = [\mathbf{L}] = \operatorname{diag}\{\dot{\lambda}/\lambda, -\dot{\lambda}/(2\lambda), -\dot{\lambda}/(2\lambda)\}, \tag{63}$$

$$I_1(\mathbf{B}) = \lambda^2 + 2/\lambda, \quad I_2(\mathbf{B}) = 2\lambda + 1/\lambda^2, \quad I_3(\mathbf{B}) = 1, \quad I_2(\mathbf{D}) = -3\dot{\lambda}^2/4\lambda^2. \tag{64}$$

Thus, the Cauchy stress components are found by (34) and are given as:

$$T_{11} = -p + \beta_1\lambda^2 + \beta_{-1}\lambda^{-2} + 2\eta\dot{\lambda}/\lambda - \frac{1}{\phi_1}(\dot{T}_{11} + 2\dot{\lambda}/\lambda T_{11}), \tag{65}$$

$$T_{22} = -p + \beta_1\lambda^{-1} + \beta_{-1}\lambda - \eta\dot{\lambda}/\lambda - \frac{1}{\phi_1}(\dot{T}_{22} - \dot{\lambda}/\lambda T_{22}). \tag{66}$$

Bearing in mind the stress-free lateral surface condition, we find $T_{22} = 0$; hence, (66) yields

$$-p = -\beta_1\lambda^{-1} - \beta_{-1}\lambda + \eta\dot{\lambda}/\lambda, \tag{67}$$

and (65) reduces to

$$\dot{T}_{11} + (2\dot{\lambda}/\lambda + \phi_1)T_{11} = \phi_1[\beta_1(\lambda^2 - \lambda^{-1}) + \beta_{-1}(\lambda^{-2} - \lambda) + 3\eta\dot{\lambda}/\lambda]. \tag{68}$$

We recall that creep in simple extension is a growth process of stretch $\lambda(t)$ under constant stress $T_{11} = \hat{T}_{11}$, say. If this constant stress is added when $t = 0$, we expect that λ will increase to the equilibrium state defined by $\dot{\lambda}(t) \to 0$ and $\lambda(t) \to \lambda_s$ as $t \to \infty$. By (68), the static stretch λ_s is related to $\hat{T}_{11}$ through

$$\hat{T}_{11} = (\lambda_s\hat{\beta}_1 - \hat{\beta}_{-1})(\lambda_s - \lambda_s^{-2}), \tag{69}$$

where $\hat{\beta}_1$ and $\hat{\beta}_{-1}$ are functions of λ_s alone and hence are constants independent of time t. The equation of creep under constant stress $\hat{T}_{11}$ may now be obtained from (68) and (69) and is given by

$$[2(\lambda_s\hat{\beta}_1 - \hat{\beta}_{-1})(\lambda_s - \lambda_s^{-2}) - 3\eta\phi_1]\frac{\dot{\lambda}}{\lambda} = -\phi_1[(\lambda_s\hat{\beta}_1 - \hat{\beta}_{-1})(\lambda_s - \lambda_s^{-2}) - (\lambda\beta_1 - \beta_{-1})(\lambda - \lambda^{-2})]. \tag{70}$$

Recovery is a decay process marked by decreasing stretch from an initially stretched state following a sudden reduction in the applied force. If the process starts from the equilibrium state determined by (69) and the force is reduced to zero, we shall see that the recovery stretch $\lambda(t)$ decreases from λ_s to zero. The equation of recovery is found from (68) as:

$$-3\eta\frac{\dot{\lambda}}{\lambda} = (\lambda\beta_1 - \beta_{-1})(\lambda - \lambda^{-2}). \tag{71}$$

The solution to (70) depends on the response functions of β_1 and β_{-1}. By (10), eqn (70) for the neo-Hookean material is reduced to

$$-\left[3\frac{\eta}{G} - \frac{2}{\phi_1}(\lambda_s^2 - \lambda_s^{-1})\right]\frac{d\lambda}{dt} = \lambda^3 + (\lambda_s^{-1} - \lambda_s^2)\lambda - 1. \tag{72}$$

This equation may be written as:

$$-t_r\dot{\lambda} = \lambda^3 + (\lambda_s^{-1} - \lambda_s^2)\lambda - 1, \tag{73}$$

where the coefficient of $\dot{\lambda}$, which is the retardation time t_r in this problem, is given by

$$t_r = 3\eta/G - 2(\lambda_s^2 - \lambda_s^{-1})/\phi_1. \tag{74}$$

We notice that in eqn (73), material constant ϕ_1 appears only in t_r. We recall (34) and realize than when $\phi_1 \to \infty$, all equations of the rate type material reduce to those of the differential type material. Hence, the first term in (74) is the retardation time of the same problem for the differential type theory. Hence, by reviewing creep equation (73) we find that the only difference between these two theories for creep in simple extension is the retardation time t_r, i.e. the creeping speed. Since ϕ_1 is positive, we find from (74) that the rate type theory predicts faster creeping speed in extension ($\lambda_s > 1$) and slower creeping speed in compression ($\lambda_s < 1$). The difference in creeping speed disappears when $\phi_1 \to \infty$.

We suppose that the creep of stretch λ starts from the unstretched state under the instantaneous stress $\hat{T}_{11}$. The asymptotic equilibrium stretch λ_s is then a factor of the cubic polynomial in (73) and the initial condition for (73) is $\lambda = 1$ when $t = 0$. The closed form solution to (73) for the given initial condition under different values of λ_s is given by:

Case 1: $\lambda_s > 4^{1/3}$

$$\frac{t}{t_r} = -\frac{1}{2\lambda_s^2 + \lambda_s^{-1}}\ln\frac{(\lambda_s - \lambda)\sqrt{1 + \lambda_s + \lambda_s^{-1}}}{(\lambda_s - 1)\sqrt{\lambda^2 + \lambda_s\lambda + \lambda_s^{-1}}} + \frac{3\lambda_s}{2(2\lambda_s^2 + \lambda_s^{-1})\sqrt{\lambda_s^2 - 4\lambda_s^{-1}}}\ln\frac{(2\lambda + \lambda_s - \sqrt{\lambda_s^2 - 4\lambda_s^{-1}})(2 + \lambda_s + \sqrt{\lambda_s^2 - 4\lambda_s^{-1}})}{(2\lambda + \lambda_s + \sqrt{\lambda_s^2 - 4\lambda_s^{-1}})(2 + \lambda_s - \sqrt{\lambda_s^2 - 4\lambda_s^{-1}})}, \tag{75}$$

Case 2: $\lambda_s = 4^{1/3}$

$$\frac{t}{t_r} = -\frac{1}{2\lambda_s^2 + \lambda_s^{-1}}\ln\frac{(\lambda_s - \lambda)\sqrt{1 + \lambda_s + \lambda_s^{-1}}}{(\lambda_s - 1)\sqrt{\lambda^2 + \lambda_s\lambda + \lambda_s^{-1}}} + \frac{6\lambda_s^2(\lambda - 1)}{(2 + \lambda_s)(2\lambda_s^3 + 1)(\lambda_s + 2\lambda)}, \tag{76}$$

Case 3: $1 < \lambda_s < 4^{1/3}$

$$\frac{t}{t_r} = -\frac{1}{2\lambda_s^2 + \lambda_s^{-1}}\ln\frac{(\lambda_s - \lambda)\sqrt{1 + \lambda_s + \lambda_s^{-1}}}{(\lambda_s - 1)\sqrt{\lambda^2 + \lambda_s\lambda + \lambda_s^{-1}}} + \frac{3\lambda_s}{(2\lambda_s^2 + \lambda_s^{-1})\sqrt{4\lambda_s^{-1} - \lambda_s^2}}\tan^{-1}\frac{\sqrt{4\lambda_s^{-1} - \lambda_s^2}(\lambda - 1)}{\lambda_s + 2\lambda_s^{-1} + (2 + \lambda_s)\lambda}, \tag{77}$$

Case 4: $0 < \lambda_s < 1$

$$\frac{t}{t_r} = -\frac{1}{2\lambda_s^2+\lambda_s^{-1}} \ln \frac{(\lambda-\lambda_s)\sqrt{1+\lambda_s+\lambda_s^{-1}}}{(1-\lambda_s)\sqrt{\lambda^2+\lambda_s\lambda+\lambda_s^{-1}}} + \frac{3\lambda_s}{(2\lambda_s^2+\lambda_s^{-1})\sqrt{4\lambda_s^{-1}-\lambda_s^2}} \tan^{-1} \frac{\sqrt{4\lambda_s^{-1}-\lambda_s^2}(\lambda-1)}{\lambda_s+2\lambda_s^{-1}+(2+\lambda_s)\lambda}. \tag{78}$$

All solutions show, of course, that $t = 0$ when $\lambda = 1$. By (73) we find that $\dot{\lambda}(0) = (\lambda_s^2-\lambda_s^{-1})/t_r$ initially. Hence, $\dot{\lambda}(0) > 0$ for the case of extension ($\lambda_s > 1$). The creeping stretch increases from 1 to λ_s. Similarly, for compression with $\lambda_s < 1$, we find $\dot{\lambda}(0) < 0$ and the creeping stretch decreases from 1 to λ_s. The creeping velocity $\dot{\lambda}$ for the entire creep process may be found by (73) which is

$$\dot{\lambda} = \frac{1}{t_r}(\lambda_s-\lambda)(\lambda^2+\lambda_s\lambda+\lambda_s^{-1}). \tag{79}$$

Hence, the creeping velocity is always positive for extension, negative for compression. At equilibrium state with $\lambda = \lambda_s$, we find from (79) that $\dot{\lambda} = 0$. As λ approaches its equilibrium state of λ_s, eqns (75)–(78) show that $t \to \infty$, as expected. Hence, these solutions describe the creep process from the initial undeformed state of $\lambda_0 = 1$ to its asymptotic equilibrium state of stretch λ_s. These results are demonstrated in Fig. 2 for different values of λ_s. The retardation ratio in this case is given by $(\lambda-1)/(\lambda_s-1)$ when $t = t_r$. This ratio for selected values of λ_s is listed as

$\lambda_s = 2.0$: 99.95%,

$\lambda_s = 4^{1/3}$: 99.37%,

$\lambda_s = 1.25$: 97.45%,

$\lambda_s = 0.5$: 93.78%.

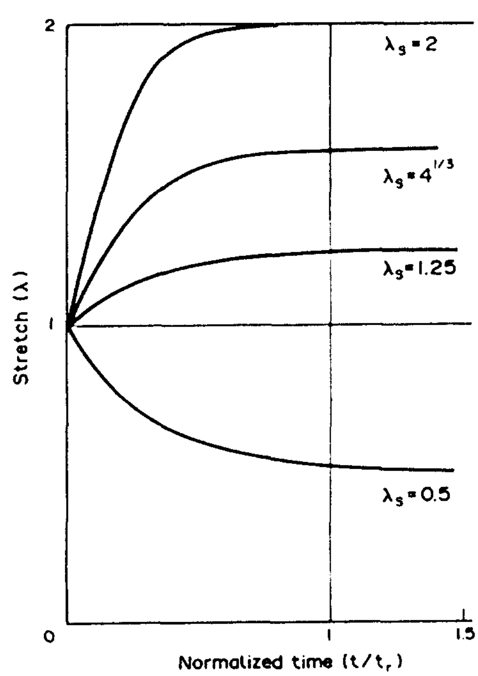

Fig. 2. Creep response of a viscoelastic neo-Hookean material in simple extension for various values of ultimate equilibrium stretch λ_s.

For example, for $\lambda_s = 1.25$, 97.45% of the total creep process is accomplished by the time $t = t_r$. This list shows that when λ_s increases, the creep process moves faster. On the other hand, bearing in mind the universal retardation ratio for simple shear deformation, the creep of simple extension moves much faster than that of simple shear.

The equation of recovery (71) for the neo-Hookean material appears in the form of

$$-t_r\dot{\lambda} = \lambda^3 - 1, \tag{80}$$

where t_r is again the retardation time and is given by

$$t_r = 3\frac{\eta}{G}. \tag{81}$$

We notice that the recovery eqns (71) and (80) contain no material constant ϕ_1. Hence, the recovery responses of the rate type material and the differential type material are the same. The recovery test hence cannot be used to distinguish these two materials.

We suppose that the recovery process starts from a certain initial stretch λ_s. By (80), the velocity of the recovery process is given by

$$\dot{\lambda} = \frac{1}{t_r}(1-\lambda^3). \tag{82}$$

Hence, the recovery velocity is always negative for extension, positive for compression. It becomes zero when $\lambda = 1$, the ultimate equilibrium state for recovery process. The closed form solutions of (80) are given by

Case 1 : $\lambda_s > 1$

$$\frac{t}{t_r} = \frac{1}{3}\ln\frac{(\lambda_s-1)\sqrt{\lambda^2+\lambda+1}}{(\lambda-1)\sqrt{\lambda_s^2+\lambda_s+1}} - \frac{\sqrt{3}}{3}\tan^{-1}\frac{\sqrt{3}(\lambda_s-\lambda)}{(2\lambda_s+1)\lambda+(\lambda_s+2)}, \tag{83}$$

Case 2 : $\lambda_s < 1$

$$\frac{t}{t_r} = \frac{1}{3}\ln\frac{(1-\lambda_s)\sqrt{\lambda^2+\lambda+1}}{(1-\lambda)\sqrt{\lambda_s^2+\lambda_s+1}} + \frac{\sqrt{3}}{3}\tan^{-1}\frac{\sqrt{3}(\lambda-\lambda_s)}{(2\lambda_s+1)\lambda+(\lambda_s+2)}. \tag{84}$$

Both solutions describe the recovery process from λ_s (as $t = 0$) to its asymptotic equilibrium state of $\lambda = 1$ (as $t \to \infty$). These results are demonstrated in Fig. 3 for different

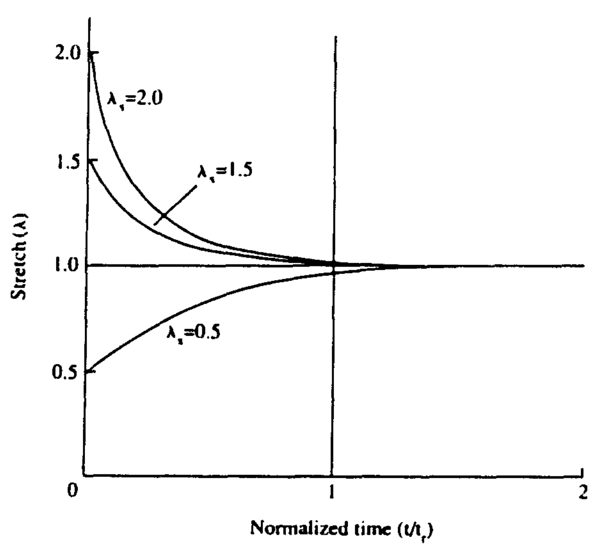

Fig. 3. Recovery response of a viscoelastic neo-Hookean material in simple extension for various values of initial stretch λ_s.

values of λ_r. The retardation ratio in this case is given by $(\lambda_r - \lambda)/(\lambda_r - 1)$ as $t = t_r$. This ratio for selected values of λ_r is given by

$$\begin{aligned} \lambda_r &= 2.0: \quad 97.6\%, \\ \lambda_r &= 1.5: \quad 96.7\%, \\ \lambda_r &= 0.5: \quad 91.3\%. \end{aligned} \tag{85}$$

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